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SOME RESULTS ON FIXED POINT THEOREMS IN COMPLETE DISLOCATED QUASI-METRIC SPACE

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Abstract

In this research paper, we have proved uniqueness and existence of common fixed point theorems for complete dislocated quasi-metric space (DQMS). Our results generalizes fixed point results in complete DQMSin existing research.

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1. Introduction

Fixed point theorem given by S. Banach [3] appeared in 1922, which is useful to solve the existence of solutions for the various nonlinear problems which used to arise in the various fields of sciences like biological, physical and social sciences. In 1994, S. Abramski et al. gave few facts pertaining the dislocated metric (DM). The notion of DM space was initiated by P. Hitzler et al. [4] in 2000 and gave some extensions of Banach contraction principle in complete dislocated metrics space (DMS). Zeyada et al. [8] in 2006 gave the generalization of results given by Hitzler and Seda in DQMS. Aage and Salunkein [1] gave some results in DQMS in 2008. For more results about dislocated quasi-metric space one can refer [5, 6]. In this paper, we will use the acronym DQMS for dislocated quasi-metric space.

2. Preliminaries

Definition 2.1 [7]. For $X \neq \emptyset$ consider the map $d : X \times X \to [0, \infty)$ which satisfies:

$(dq_1) \cdot d(x, y) = d(y, x) = 0 \Rightarrow x = y$

$(dq_2) \cdot d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$

Here d is called dislocated quasi-metric on X and (X, d) is DQMS.

Definition 2.2 [7]. In a DQMS (X, d), sequence $\{x_n\}$ converges to z, if $\lim_{n\to\infty} d(x_n, z) = 0 = \lim_{n\to\infty} d(z, x_n)$.

Definition 2.3 [7]. In a DQMS (X, d), $\{x_n\}$ is said to be Cauchy if for a given $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that, $\forall m, n \geq n_0$, $d(x_n, x_m) < \epsilon$ or $d(x_m, x_n) < \epsilon$.

Definition 2.4 [7]. ADQMS (X, d) is complete, if each Cauchy sequence in X is dq-convergent to a point in X.

Definition 2.5 [7]. If (X, d) be a DQMS. Then $f : X \to X$ is contraction if $\exists\, 0 \leq \alpha < 1$ such that for all $x, y \in X$, $d(fx, fy) \leq \alpha d(x, y)$.

Theorem 2.6 [8]. *Let (X, d) be a complete DQMS and let $T : X \to X$ be continuous contraction function. Then in X there exists a unique fixed point of T.*

Theorem 2.7 [1]. *Suppose* (X, d) *is complete DQMS and* $T : X \to X$ *be a continuous se lf mapping such that* $\forall\ x, y \in X, d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)], 0 \leq \beta \leq \frac{1}{2}$.

Then fixed point of T is unique.

3. Main Results

Theorem 3.1. *In complete DQMS* (X, d), *suppose* $T : X \to X$ *be continuous mapping*, *such that*

$$\forall\ x, y \in X, 0 \leq \alpha < \frac{1}{2}, 0 \leq \beta < \frac{1}{2} \text{ and } 4\alpha + \beta < 1,$$

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)] + \beta \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} \tag{3.1}$$

Then $\exists!\ u \in X$ *such that* $T(u) = u$.

Proof. Suppose $x_0 \in X$ and $\{x_n\}$ a sequence in X such that $Tx_n = x_{n+1}$.

Consider,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha[d(Tx_{n-1}, Tx_n) + d(x_n, x_{n+1})] +$$

$$\beta \frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)}$$

$$= \alpha[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + \beta \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)}$$

$$\leq \alpha[d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})] + \beta d(x_n, x_{n+1}) + \alpha[d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})]$$

(by triangle inequality and $d(x_n, x_n) = 0$)

$$\Rightarrow (1 - 2\alpha - \beta) d(x_n, x_{n+1}) \leq 2\alpha d(x_{n-1}, x_n)$$

i.e. $d(x_n, x_{n+1}) \leq \frac{2\alpha}{1 - (2\alpha + \beta)} d(x_{n-1}, x_n) = \lambda d(x_{n-1}, x_n),$ where $\lambda = \frac{2\alpha}{1 - (2\alpha + \beta)} < 1,$

[since $0 \leq 4\alpha + \beta < 1 \Rightarrow 2\alpha + (2\alpha + \beta) < 1$

i.e. $2\alpha < 1 - (2\alpha + \beta) \Rightarrow \frac{2\alpha}{1 - (2\alpha + \beta)} < 1$]

Similarly, $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$

$$d(x_n, x_{n+1}) \leq \lambda^2 d(x_{n-2}, x_{n-1})$$

Continuing in this way, we get

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$$

Now, for any $m, n \in X$ with $m > n$, using the triangle inequality, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)$$

$$\leq \lambda^n d(x_0, x_1) \leq \lambda^{n+1} d(x_0, x_1) + \ldots + \lambda^{m-1} d(x_0, x_1)$$

$$\leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1} = \frac{\lambda^n}{1 - \lambda} d(x_0, x_1), \text{ since } \lambda < 1$$

For any $\epsilon > 0$, choose $N \in \mathbb{N}$ with $\frac{\lambda^N}{1 - \lambda} d(x_0, x_1) < \epsilon$.

For any $m, n \geq N$, we have

$$d(x_n, x_m) \leq \frac{\lambda^n}{1 - \lambda} d(x_0, x_1) \leq \frac{\lambda^N}{1 - \lambda} d(x_0, x_1) < \epsilon$$

$\Rightarrow \{x_n\}$ is Cauchy in (X, d). Therefore $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$ $\exists u \in X$ with $\lim_{n \to \infty} x_n = u$.

$\because T$ is continuous,

$$\because Tu = T(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = u$$

Hence, $T(u) = u$.

Uniqueness of fixed point:

Suppose that for $u, v \in X$ with $u \neq v$ and $T(u) = u$, $T(v) = v$.

We will show that $d(u, u) = 0$ and $d(v, v) = 0$. If $d(u, u) > 0$, then by equation (3.1), we have $d(u, u) = d(Tu, Tu) \le \alpha[d(u, Tu) + d(u, Tu)] + \beta \frac{d(u, Tu)[1 + d(u, Tu)]}{1 + d(u, u)} \le (2\alpha + \beta) d(u, u) < d(u, u)$ since $0 \le 2\alpha + \beta < 1$ which is a contraction. Hence $d(u, u) = 0$.

Similarly, one can show that $d(v, v) = 0$.

Now, we will show that $d(u, v) = d(v, u) = 0$

From equation (3.1), we have

$$d(u, v) = d(Tu, Tv) \le \alpha[d(u, Tv) + d(v, Tu)] + \beta \frac{d(v, Tv)[1 + d(u, Tu)]}{1 + d(u, v)}$$

$$\le \alpha[d(u, v) + d(v, u)] + \beta \frac{d(v, v)[1 + d(u, u)]}{1 + d(u, v)} \tag{3.2}$$

and

$$d(v, u) = d(Tu, Tv) \le \alpha[d(v, Tu) + d(u, Tv)] + \beta \frac{d(u, Tu)[1 + d(v, Tv)]}{1 + d(v, u)}$$

$$\le \alpha[d(v, u) + d(u, v)] + \beta \frac{d(u, u)[1 + d(v, v)]}{1 + d(v, u)}. \tag{3.3}$$

From equations (3.2) and (3.3), we get,

$$| d(u, v) - d(v, u) | = 0 \Rightarrow d(u, v) = d(v, u). \tag{3.4}$$

Using equation (3.4) in equations (3.2) and (3.3), we get $d(u, v) = 0 = d(v, u)$

Therefore, by definition of DQMS, $u = v$, which is a contradiction.

Hence, $\exists!\, u \in X$ with $T(u) = u$.

Theorem 3.2. *For a complete DQMS* (X, d), *let* $T : X \to X$ *be a continuous mapping, such that* $\forall\, x, y \in X$, $0 \le \alpha_1, \alpha_2, \alpha_3 < \frac{1}{2}$, $2\alpha_1 + 4\alpha_2 + \alpha_3 < 1$,

$$d(Tx, Ty) \leq \alpha_1[d(x, Tx) + d(y, Ty)] + \alpha_2[d(x, Ty) + d(y, Tx)] + \alpha_3 \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} \tag{3.5}$$

Then $\exists!\, u \in X$ *with* $T(u) = u$.

Proof. Suppose $x_0 \in X$ and $\{x_n\}$ be sequence in X such that $Tx_n = x_{n+1}$.

Consider, $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$$\leq \alpha_1[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + \alpha_2[d(x_{n-1}, Tx_n) + d(x_n, Tx_n)] + \alpha_3 \frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)}$$

$$= \alpha_1[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \alpha_2[d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})] + \alpha_3 \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)}$$

$$\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_{n-1}, x_{n+1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_2 d(x_{n-1}, x_n) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_n, x_{n+1})$$

(by triangle inequality)

$$\Rightarrow (1 - \alpha_1 - 2\alpha_2 - \alpha_3) d(x_n, x_{n+1}) \leq (\alpha_1 + 2\alpha_2) d(x_{n-1}, x_n)$$

i.e. $d(x_n, x_{n+1}) \leq \frac{(\alpha_1 + 2\alpha_2)}{(\alpha_1 + 2\alpha_2 + \alpha_3)} d(x_{n-1}, x_n) = \lambda d(x_{n-1}, x_n)$, where $\lambda = \frac{(\alpha_1 + 2\alpha_2)}{1 - (\alpha_1 + 2\alpha_2 + \alpha_3)} < 1$

[since $0 \leq 2\alpha_1 + 4\alpha_2 + \alpha_3 < 1 < 1 \Rightarrow (\alpha_1 + 2\alpha_2) + (\alpha_1 + 2\alpha_2 + \alpha_3) < 1$

i.e. $(\alpha_1 + 2\alpha_2) < 1 - (\alpha_1 + 2\alpha_2 + \alpha_3) \Rightarrow \frac{(\alpha_1 + 2\alpha_2)}{1 - (\alpha_1 + 2\alpha_2 + \alpha_3)} < 1$]

Similarly, $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$

$$d(x_n, x_{n+1}) \leq \lambda^2 d(x_{n-2}, x_{n-1})$$

Continuing in this way, we get

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$$

Now, for any $m, n \in X$ with $m > n$, using the triangle inequality, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)$$

$$\leq \lambda^n d(x_0, x_1) \leq \lambda^{n+1} d(x_0, x_1) + \ldots + \lambda^{m-1} d(x_0, x_1)$$

$$\leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1} = \frac{\lambda^n}{1-\lambda} d(x_0, x_1), \text{ since } \lambda < 1$$

For any $\epsilon > 0$, choose $N \in \mathbb{N}$ with $\frac{\lambda^n}{1-\lambda} d(x_0, x_1) < \epsilon$.

Then for $m, n \geq N$, we have

$$d(x_n, x_m) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1) \leq \frac{\lambda^N}{1-\lambda} d(x_0, x_1) < \epsilon$$

Similarly, it is easy to show $d(x_m, x_n) \leq \epsilon$

This shows that, $\{x_n\}$ is a Cauchy in DQMS (X, d). Therefore $\exists u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Since T is continuous, so we have

$$\because Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = u$$

Hence, $T(u) = u$.

Uniqueness of fixed point:

For $u, v \in X$ with $u \neq v$ and $T(u) = u, T(v) = v$.

We will show that $d(u, u) = 0$ and $d(v, v) = 0$. If $d(u, u) > 0$, then by equation (3.5), we have $d(u, u) = d(Tu, Tu) \leq \alpha_1[d(u, Tu) + d(u, Tu)] +$

$$\alpha_2[d(u, Tu) + d(u, Tu)] + \alpha_3 \frac{d(u, Tu)[1 + d(u, Tu)]}{1 + d(u, u)}$$

$$\leq (2\alpha_1 + 2\alpha_2 + \alpha_3) d(u, u) < d(u, u)$$

since $0 \leq (2\alpha_1 + 2\alpha_2 + \alpha_3) < 1$, which is a contraction. Hence $d(u, u) = 0$.

Similarly, one can show that $d(v, v) = 0$

Now, we will show that $d(u, v) = d(v, u) = 0$

From equation (3.5), we have

$$d(u, v) = d(Tu, Tv)$$

$$\leq \alpha_1[d(u, Tu) + d(v, Tv)] + \alpha_2[d(u, Tv) + d(v, Tu)]$$

$$+ \alpha_3 \frac{d(v, Tv)[1 + d(u, Tu)]}{1 + d(u, v)}$$

$$= \alpha_1[d(u, u) + d(v, v)] + \alpha_2[d(u, v) + d(v, u)] + \alpha_3 \frac{d(v, v)[1 + d(u, u)]}{1 + d(u, v)} \quad (3.6)$$

And

$$d(v, u) = d(Tv, Tu)$$

$$\leq \alpha_1[d(v, Tv) + d(u, Tu)] + \alpha_2[d(v, Tu) + d(u, Tv)]$$

$$+ \alpha_3 \frac{d(u, Tu)[1 + d(v, Tv)]}{1 + d(v, u)}$$

$$\leq \alpha_1[d(v, v) + d(u, u)] + \alpha_2[d(v, u) + d(u, v)] + \alpha_3 \frac{d(u, u)[1 + d(v, v)]}{1 + d(v, u)} \quad (3.7)$$

From equations (3.6) and (3.7), we get

$$| d(u, v) - d(v, u) | = 0 \Rightarrow d(u, v) = d(v, u) \quad (3.8)$$

Using equation (3.8) in equations (3.6) and (3.7), we get $d(u, v) = 0 = d(v, u)$

Therefore, by definition of DQMS, $u = v$, which is not true.

$\therefore \exists!\, u \in X$ such that $T(u) = u$.

4. Conclusion

Here we proved two fixed point theorems for self-continuous function in complete DQMS which combines, generalizes a number of familiar results in the history of fixed point theory.

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